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De multiplicatione angulorum per factores expedienda

Leonhard Euler

DE MVLTIPLICATIONE ANGVLORVM PER FACTORES EXPEDIENDA.

Auctore
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§. 1.

Denotet Φ angulum quemcunque propositum, sitque $n\Phi$ eius multiplum, cuius tam sinum quam cosinum per factores exprimere oporteat. Ad hoc praestandum in subsidium vocetur formula imaginaria $u = \cos.\Phi + \sqrt{-1}.\sin.\Phi$, eritque $\frac{1}{u} = u^{-1} = \cos.\Phi - \sqrt{-1}.\sin.\Phi$, vnde ergo sequitur fore $u + u^{-1} = 2\cos.\Phi$ et $u - u^{-1} = 2\sqrt{-1}.\sin.\Phi$. Constat autem similem fore rationem omnium potestatum huius formulae, siquidem erit $u^n = \cos.n\Phi + \sqrt{-1}\sin.n\Phi$ et $u^{-n} = \cos.n\Phi - \sqrt{-1}\sin.n\Phi$, atque hinc colligimus

$$u^n + u^{-n} = 2\cos.n\Phi \text{ et } u^n - u^{-n} = 2\sqrt{-1}\sin.n\Phi.$$

Hinc igitur sequitur fore

$$\cos.n\Phi = \tfrac{1}{2}(u^n + u^{-n}) \text{ et } \sin.n\Phi = \frac{1}{2\sqrt{-1}}(u^n - u^{-n}).$$

Per huiusmodi ergo formulas etiam tangentem et cotangentem, item secantem et cosecantem anguli multipli $n\Phi$ exprimere licebit, ita vt sufficiat eius tantum sinum et cosinum in factores resoluisse, quorum vtrumque hic seorsim euoluamus.

I. Resolutio cosinus anguli multipli $n\phi$ in factores.

§. 2. Cum sit, vt modo vidimus,

$$\cos. n\phi = \tfrac{1}{2}(u^n + u^{-n}),$$

totum negotium huc redit, vt formula $u^n + u^{-n}$ in suos factores resoluatur. Facile autem perspicitur, singulos huius formulae factores talem formam esse habituros: $u - 2\cos. \omega + u^{-1}$, ac numerum talium factorum esse debere $= n$; quandoquidem si n huiusmodi factores in se inuicem multiplicentur, summa potestas positiua ipsius u vtique prodibit u^n, summa autem potestas negatiua $= u^{-n}$; ex quo necesse est, vt omnes potestates intermediae ipsius u se mutuo destruant. Ex hac igitur conditione singulos angulos ω definire oportebit, id quod vtique insignem laborem postularet; vnde hanc inuestigationem ex ipsa indole factorum deriuemus.

§. 3. Hic autem perspicuum est, si haec formula: $u - 2\cos. \omega + u^{-1}$, fuerit factor formae propositae $u^n + u^{-n}$, angulum ω ita comparatum esse debere, vt si statueretur iste factor $u - 2\cos. \omega + u^{-1} = 0$, tum etiam ipsa forma proposita $u^n + u^{-n}$ ad nihilum redigeretur. Neque vero hic putandum est, istam formulam reuera nihilo aequalem supponi, sed tantum hypothetice asseritur, si ista formula, esset $= 0$, etiamsi id fieri nequeat, tum etiam ipsam formam propositam in nihilum esse abituram; quemadmodum scilicet nouimus, formulae $1 - xx$ factorem esse $1 - x$, propterea quod posito $1 - x = 0$, siue $x = 1$, etiam fiat $1 - xx = 0$, etiamsi quantitas x fortasse nunquam vnitati fiat aequalis.

§. 4. Iam vero si esset $u - 2\cos. \omega + u^{-1} = 0$, inde foret $u = \cos. \omega + \sqrt{-1}\,\sin. \omega$, quamobrem hic valor, in formam

mam $u^n + u^{-n}$ introductus, eam ad nihilum redigere deberet; tum autem foret, vt supra notauimus:

$$u^n = \text{cos}.\, n\,\omega + \sqrt{-1}\ \text{sin}.\, n\,\omega \quad \text{et}$$
$$u^{-n} = \text{cos}.\, n\,\omega - \sqrt{-1}\ \text{sin}.\, n\,\omega,$$

sicque forma proposita fiet $= 2\ \text{cos}.\, n\,\omega$, quae ergo quantitas necessario euanescere debet, quia aliter formula assumta factor formae propositae esse non posset.

§. 5. Hinc igitur patet angulum ω ita accipi debere, vt eius multipli $n\,\omega$ cosinus euanescat; vnde manifestum est, si ϱ designet angulum rectum, tum isti conditioni satisfieri, si statuatur vel $n\,\omega = \varrho$; vel $n\,\omega = 3\,\varrho$; vel $n\,\omega = 5\,\varrho$; vel $n\,\omega = 7\,\varrho$; etc. ex quo valores idonei pro angulo quaesito ω erunt sequentes: 1°. $\frac{\varrho}{n}$; 2°. $\frac{3\varrho}{n}$; 3°. $\frac{5\varrho}{n}$; 4°. $\frac{7\varrho}{n}$; etc. et in genere $\frac{i\varrho}{n}$, denotante i numerum imparem quemcunque; ita vt horum valorum numerus reuera sit infinitus, cum tamen pro nostro instituto tantum n valoribus diuersis indigeamus.

§. 6. Quanquam autem numerus valorum pro ω inuentorum sit infinitus, tamen inter eos innumerabiles dantur, qui pro cos. ω eundem valorem producunt. Cum enim omnibus istis angulis: ψ, $4\varrho + \psi$, $8\varrho + \psi$, etc. idem cosinus conueniat, etiam omnes isti anguli: $\frac{\varrho}{n}$, $\frac{(4n+1)\varrho}{n}$, $\frac{(8n+1)\varrho}{n}$, etc. communi gaudebunt cosinu, similique modo etiam omnes hi anguli: $\frac{3\varrho}{n}$, $\frac{(4n+3)\varrho}{n}$, $\frac{(8n+3)\varrho}{n}$, etc., tum vero etiam isti: $\frac{5\varrho}{n}$, $\frac{(4n+5)\varrho}{n}$, $\frac{(8n+5)\varrho}{n}$, etc. communi cosinu erunt praediti; vnde ex quolibet ordine vnicum tantum valorem, eumque simplicissimum, sumi conueniet, donec eorum numerus fiat $= n$.

§. 7. Ex his igitur colligimus, omnes valores idoneos pro angulo ω assumendos ordine ita progredi:

$$\begin{array}{cccc} 1. & 2. & 3. & 4. \\ \frac{\varrho}{n}; & \frac{3\varrho}{n}; & \frac{5\varrho}{n}; & \frac{7\varrho}{n}; \text{ etc.} \end{array}$$

quorum numerus cum debeat esse $= n$, vltimus eorum erit $\frac{(2n-1)\varrho}{n}$. Si enim vlterius progredi vellemus, ad eiusmodi angulos perueniremus, quorum cosinus iam in praecedentibus occurrerent. Anguli enim proxime sequentis $\frac{(2n+1)\varrho}{n}$ cosinus foret ipsius vltimi $\frac{(2n-1)\varrho}{n}$ cosinui aequalis: in genere enim hi duo anguli ψ et $4\varrho - \psi$ communem habent cosinum; vnde si fuerit $\psi = \frac{(2n+1)\varrho}{n}$, erit $4\varrho - \psi = \frac{(2n-1)\varrho}{n}$, qui est terminus vltimus. Simili modo sequentium secundus $\frac{(2n+3)\varrho}{n}$, eundem habet cosinum quem angulus $\frac{(2n-3)\varrho}{n}$ habet, qui in nostro ordine est penultimus. Eodem modo sequentium tertius $\frac{(2n+5)\varrho}{n}$ cum nostro antepenultimo $\frac{(2n-5)\varrho}{n}$ communem habebit cosinum; quod idem de reliquis sequentibus est tenendum, quippe quorum omnium cosinus iam in nostro ordine occurrunt; vnde patet omnes valores idoneos anguli ω, quorum numerus est $= n$, contineri in hac serie:

$$\begin{array}{cccccccccc} 1. & 2. & 3. & 4. & . & . & . & . & . & n \\ \frac{\varrho}{n}; & \frac{3\varrho}{n}; & \frac{5\varrho}{n}; & \frac{7\varrho}{n}; & . & . & . & . & & \frac{(2n-1)\varrho}{n}. \end{array}$$

§. 8. Cum igitur, si pro ω accipiatur valor quicunque huius progressionis, ista formula $u - 2\,\text{cos.}\,\omega + u^{-1}$ certe sit factor formae $u^n + u^{-n}$, restituamus nunc valorem initio assumtum $u = \text{cos.}\,\Phi + \sqrt{-1}\,.\,\text{sin.}\,\Phi$, et quia hinc $u + u^{-1} = 2\,\text{cos.}\,\Phi$, huius formae $u^n + u^{-n}$ factor quicunque erit $2\,(\text{cos.}\,\Phi - \text{cos.}\,\omega)$, quare si loco ω successiue omnes eius valores scribamus, omnes factores obtinebimus, quorum numerus cum sit $= n$, forma nostra $u^n + u^{-n}$ aequabitur huic producto ex n factoribus constanti:

2^n

$$2^n (\cos.\Phi - \cos.\tfrac{\varrho}{n}) (\cos.\Phi - \cos.\tfrac{3\varrho}{n}) (\cos.\Phi - \cos.\tfrac{5\varrho}{n}) \times$$
$$\times (\cos.\Phi - \cos.\tfrac{7\varrho}{n}) \ldots\ldots (\cos.\Phi - \cos.\tfrac{(2n-1)\varrho}{n}).$$

§. 9. Cum igitur ſit $\cos. n\Phi = \frac{1}{2}(u^n + u^{-n})$, ſi hoc productum ſubſtituamus, coſinus anguli $n\Phi$ ſequenti modo per productum ex n factoribus compoſitum exprimetur:

$$\cos. n\Phi = 2^{n-1} (\cos.\Phi - \cos.\tfrac{\varrho}{n}) (\cos.\Phi - \cos.\tfrac{3\varrho}{n}) \times$$
$$\times (\cos.\Phi - \cos.\tfrac{5\varrho}{n}) \ldots\ldots (\cos.\Phi - \cos.\tfrac{(2n-1)\varrho}{n}),$$

vnde ſequentes deducimus reſolutiones ſpeciales, dum loco n ordine numeros 1, 2, 3, 4, etc. aſſumimus:

$$\cos. 1\Phi = \cos.\Phi - \cos.\tfrac{\varrho}{1} = \cos.\Phi.$$
$$\cos. 2\Phi = 2 (\cos.\Phi - \cos.\tfrac{1}{2}\varrho) (\cos.\Phi - \cos.\tfrac{3}{2}\varrho).$$
$$\cos. 3\Phi = 4 (\cos.\Phi - \cos.\tfrac{1}{3}\varrho) (\cos.\Phi - 0) (\cos.\Phi - \cos.\tfrac{5}{3}\varrho).$$
$$\cos. 4\Phi = 8 (\cos.\Phi - \cos.\tfrac{1}{4}\varrho) (\cos.\Phi - \cos.\tfrac{3}{4}\varrho) (\cos.\Phi - \cos.\tfrac{5}{4}\varrho) \times$$
$$\times (\cos.\Phi - \cos.\tfrac{7}{4}\varrho).$$
$$\cos. 5\Phi = 16 (\cos.\Phi - \cos.\tfrac{1}{5}\varrho) (\cos.\Phi - \cos.\tfrac{3}{5}\varrho) (\cos.\Phi - 0) \times$$
$$\times (\cos.\Phi - \cos.\tfrac{7}{5}\varrho) (\cos.\Phi - \cos.\tfrac{9}{5}\varrho)$$
$$\cos. 6\Phi = 32 (\cos.\Phi - \cos.\tfrac{1}{6}\varrho) (\cos.\Phi - \cos.\tfrac{1}{2}\varrho) (\cos.\Phi - \cos.\tfrac{5}{6}\varrho) \times$$
$$\times (\cos.\Phi - \cos.\tfrac{7}{6}\varrho) (\cos.\Phi - \cos.\tfrac{3}{2}\varrho) (\cos.\Phi - \cos.\tfrac{11}{6}\varrho).$$

etc. etc.

§. 10. Quodſi omnes iſtos factores attentius conſideremus, reperiemus binos factores ab extremis aeque diſtantes commode in vnum contrahi poſſe. Cum enim duorum angulorum, quorum ſumma eſt $180° = 2\varrho$, coſinus ſint aequales, ſed contrario ſigno affecti, ob angulum $\frac{(2n-1)\varrho}{n} = 2\varrho - \frac{\varrho}{n}$, eius coſinus erit $= -\cos.\frac{\varrho}{n}$, ideoque vltimus factor $= \cos.\Phi$

$+ \cos.$

$+ \operatorname{cos.} \frac{\varrho}{n}$, qui ductus in primum praebet productum $\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \frac{\varrho}{n})^2$. Simili modo pro penultimo factore, ob $\frac{(2n-3)\varrho}{n} = 2\varrho - \frac{3\varrho}{n}$, huius anguli cosinus erit $= - \operatorname{cos.} \frac{3\varrho}{n}$, ideoque ipse factor penultimus $= \operatorname{cos.} \Phi + \operatorname{cos.} \frac{3\varrho}{n}$, qui ductus in secundum dabit productum $\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \frac{3\varrho}{n})^2$. Eodem modo factor tertius cum antepenultimo coniunctus dabit hoc productum $\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \frac{5\varrho}{n})^2$. Hoc igitur modo numerus factorum ad semissem reducetur, si numerus n fuerit par, sin autem fuerit impar, facta hac contractione solitarius relinquetur terminus medius, qui semper erit $\operatorname{cos.} \Phi$, ob $\operatorname{cos.} \varrho = 0$.

§. 11. Quodsi ergo hoc modo resolutionem in factores exhibere velimus, ea pro casibus simplicioribus ita se habebit:

$$\operatorname{cos.} \Phi = \operatorname{cos.} \Phi.$$

$$\operatorname{cos.} 2\Phi = 2[\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{1}{2}\varrho)^2].$$

$$\operatorname{cos.} 3\Phi = 4[\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{1}{3}\varrho)^2] \operatorname{cos.} \Phi.$$

$$\operatorname{cos.} 4\Phi = 8[\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{1}{4}\varrho)^2][\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{3}{4}\varrho)^2].$$

$$\operatorname{cos.} 5\Phi = 16[\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{1}{5}\varrho)^2][\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{3}{5}\varrho)^2] \operatorname{cos.} \Phi.$$

$$\operatorname{cos.} 6\Phi = 32[\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{1}{6}\varrho)^2][\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{3}{6}\varrho)^2] \times$$
$$\times [\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{5}{6}\varrho)^2].$$

$$\operatorname{cos.} 7\Phi = 64[\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{1}{7}\varrho)^2][\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{3}{7}\varrho)^2] \times$$
$$\times [\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{5}{7}\varrho)^2] \operatorname{cos.} \Phi.$$

$$\operatorname{cos.} 8\Phi = 128[\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{1}{8}\varrho)^2][\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{3}{8}\varrho)^2] \times$$
$$\times [\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{5}{8}\varrho)^2][\operatorname{cos.} \Phi^2 - (\operatorname{cos.} \tfrac{7}{8}\varrho)^2].$$

etc. etc.

§. 12.

§. 12. Iſtos autem factores in formam adhuc conciniorem transmutare licet. Cum enim in genere ſit

$$\cos.\psi^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos.2\psi, \text{ erit}$$

$$\cos.\phi^2 - \cos.\psi^2 = \tfrac{1}{2}\cos.2\phi - \tfrac{1}{2}\cos.2\psi;$$

tum vero etiam conſtat eſſe in genere

$$\cos.2\phi - \cos.2\psi = 2\sin.(\phi + \psi)\sin.(\psi - \phi),$$

conſequenter

$$\cos.\phi^2 - \cos.\psi^2 = \sin.(\phi + \psi)\sin.(\psi - \phi).$$

§. 13. Quodſi ergo iſta reductione in ſuperioribus formulis vtamur, ſinguli factores ad ſimplices ſinus reuocabuntur; habebimus enim:

$$\cos.\phi = \cos.\phi.$$

$$\cos.2\phi = 2\sin.(\tfrac{1}{2}\varrho + \phi)\sin.(\tfrac{1}{2}\varrho - \phi)$$

$$\cos.3\phi = 4\sin.(\tfrac{1}{3}\varrho + \phi)\sin.(\tfrac{1}{3}\varrho - \phi)\cos.\phi.$$

$$\cos.4\phi = 8\sin.(\tfrac{1}{4}\varrho + \phi)\sin.(\tfrac{1}{4}\varrho - \phi)\sin.(\tfrac{3}{4}\varrho + \phi)\sin.(\tfrac{3}{4}\varrho - \phi).$$

$$\cos.5\phi = 16\sin.(\tfrac{1}{5}\varrho + \phi)\sin.(\tfrac{1}{5}\varrho - \phi)\sin.(\tfrac{3}{5}\varrho + \phi)\sin.(\tfrac{3}{5}\varrho - \phi)\times$$
$$\times\cos.\phi$$

$$\cos.6\phi = 32\sin.(\tfrac{1}{6}\varrho + \phi)\sin.(\tfrac{1}{6}\varrho - \phi)\sin.(\tfrac{3}{6}\varrho + \phi)\sin.(\tfrac{3}{6}\varrho - \phi)\times$$
$$\times\sin.(\tfrac{5}{6}\varrho + \phi)\sin.(\tfrac{5}{6}\varrho - \phi).$$

etc. etc.

vnde in genere concludimus fore

$$\cos.n\phi = 2^{n-1}\sin.(\tfrac{\varrho}{n} + \phi)\sin.(\tfrac{\varrho}{n} - \phi)\sin.(\tfrac{3\varrho}{n} + \phi)\times$$
$$\times\sin.(\tfrac{3\varrho}{n} - \phi)\sin.(\tfrac{5\varrho}{n} + \phi)\sin.(\tfrac{5\varrho}{n} - \phi)\text{ etc.}$$

Vbi obſeruetur, ſi n fuerit numerus impar, tum vltimo factori accedere $\cos.\phi$. Caeterum hos factores eo vsque produci oportet, donec eorum numerus fiat $= n$.

§. 14. Si quis maluerit omnes iſtos factores etiam per coſinus exhibere, talis transformatio in promptu eſt. Cum enim ſit ſin. $\psi = $ coſ. $(\varrho - \psi)$, noſtrae reſolutiones ſequenti modo procedent:

$$\text{coſ.}\,\Phi = \text{coſ.}\,\Phi.$$
$$\text{coſ.}\,2\Phi = 2\,\text{coſ.}\,(\tfrac{1}{2}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{1}{3}\varrho + \Phi).$$
$$\text{coſ.}\,3\Phi = 4\,\text{coſ.}\,(\tfrac{2}{3}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{2}{3}\varrho + \Phi)\,\text{coſ.}\,\Phi.$$
$$\text{coſ.}\,4\Phi = 8\,\text{coſ.}\,(\tfrac{3}{4}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{3}{4}\varrho + \Phi)\,\text{coſ.}\,(\tfrac{1}{4}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{1}{4}\varrho + \Phi)$$
$$\text{coſ.}\,5\Phi = 16\,\text{coſ.}\,(\tfrac{4}{5}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{4}{5}\varrho + \Phi)\,\text{coſ.}\,(\tfrac{2}{5}\varrho - \Phi)\,\times$$
$$\times\,\text{coſ.}\,(\tfrac{2}{5}\varrho + \Phi)\,\text{coſ.}\,\Phi.$$
$$\text{coſ.}\,6\Phi = 32\,\text{coſ.}\,(\tfrac{5}{6}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{5}{6}\varrho + \Phi)\,\text{coſ.}\,(\tfrac{3}{6}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{3}{6}\varrho + \Phi)\,\times$$
$$\times\,\text{coſ.}\,(\tfrac{1}{6}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{1}{6}\varrho + \Phi).$$
$$\text{coſ.}\,7\Phi = 64\,\text{coſ.}\,(\tfrac{6}{7}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{6}{7}\varrho + \Phi)\,\text{coſ.}\,(\tfrac{4}{7}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{4}{7}\varrho + \Phi)\,\times$$
$$\times\,\text{coſ.}\,(\tfrac{2}{7}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{2}{7}\varrho + \Phi)\,\text{coſ.}\,\Phi.$$
$$\text{coſ.}\,8\Phi = 128\,\text{coſ.}\,(\tfrac{7}{8}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{7}{8}\varrho + \Phi)\,\text{coſ.}\,(\tfrac{5}{8}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{5}{8}\varrho + \Phi)\,\times$$
$$\times\,\text{coſ.}\,(\tfrac{3}{8}\varrho - \Phi)\,\text{coſ.}\,(\tfrac{3}{8}\varrho + \Phi)\,\text{coſ.}\,(\tfrac{1}{8}\varrho - \Phi)\,\times$$
$$\times\,\text{coſ.}\,(\tfrac{1}{8}\varrho + \Phi).$$

§. 15. Quo igitur ordo in his expreſſionibus clarius ob oculos ponatur, duos caſus diſtingui conueniet, quos hic ſeorſim referemus, prouti numerus n fuerit vel par vel impar.

Caſu quo $n = 2i$, erit:

$$\text{coſ.}\,2i\Phi = 2^{2i-1}\,\text{coſ.}\,(\tfrac{\varrho}{2i} + \Phi)\,\text{coſ.}\,(\tfrac{\varrho}{2i} - \Phi)\,\text{coſ.}\,(\tfrac{3\varrho}{2i} + \Phi)\,\times$$
$$\times\,\text{coſ.}\,(\tfrac{3\varrho}{2i} - \Phi)\,\text{coſ.}\,(\tfrac{5\varrho}{2i} + \Phi)\,\text{coſ.}\,(\tfrac{5\varrho}{2i} - \Phi)\,\ldots\,\times$$
$$\times\,\text{coſ.}\,(\tfrac{(2i-1)\varrho}{2i} + \Phi)\,\text{coſ.}\,(\tfrac{(2i-1)\varrho}{2i} - \Phi).$$

Caſu

Caſu quo $n = 2i + 1$, erit

$$\text{cof.}\,(2i+1)\,\Phi = 2^{2i}\,\text{cof.}\,\Phi\,\text{cof.}\,(\tfrac{2\rho}{2i+1} + \Phi) \times$$
$$\times\,\text{cof.}\,(\tfrac{2\rho}{2i+1} - \Phi)\,\text{cof.}\,(\tfrac{4\rho}{2i+1} + \Phi)\,\text{cof.}\,(\tfrac{4\rho}{2i+1} - \Phi) \times$$
$$\times\,\text{cof.}\,(\tfrac{6\rho}{2i+1} + \Phi)\,\text{cof.}\,(\tfrac{6\rho}{2i+1} - \Phi)\;\ldots\ldots\;\times$$
$$\times\,\text{cof.}\,(\tfrac{2i\rho}{2i+1} + \Phi)\,\text{cof.}\,(\tfrac{2i\rho}{2i+1} - \Phi).$$

II. Reſolutio ſinus anguli multipli $n\Phi$ in factores.

§. 16. Manentibus denominationibus initio ſtabilitis iam notauimus eſſe ſin. $n\Phi = \frac{1}{2\sqrt{-1}}(u^n - u^{-n})$; vnde nobis incumbit in factores huius formae $u^n - u^{-n}$ inquirere. Hic autem ſtatim elucet, cunctos eius factores non habere poſſe formam ſupra aſſumtam $u - 2\,\text{cof.}\,\omega + u^{-1}$, quandoquidem ex n huiusmodi factoribus maxima poteſtas negatiua proditura eſſet $+u^{-n}$, cum tamen hic ſit $-u^{-n}$; facile autem patet formae noſtrae propoſitae $u^n - u^{-n}$ vnum factorem certe eſſe $u - u^{-1}$, quandoquidem $uu - 1$ ſemper eſt factor formae $u^{2n} - 1$. Hoc igitur primo factore conſtituto, reliquorum, quorum ergo numerus erit $= 2n - 1$, forma tuto aſſumi poteſt ſupra vſurpata $u - 2\,\text{cof.}\,\omega + u^{-1}$, vbi ergo angulum ω ita comparatum eſſe oportet, vt ſi poneretur $u - 2\,\text{cof.}\,\omega + u^{-1} = 0$, id quod fit capiendo $u = \text{cof.}\,\omega + \sqrt{-1}\,.\,\text{ſin.}\,\omega$, ipſa forma propoſita $u^n - u^{-n}$ ad nihilum redigeretur.

§. 17. Supra autem iam obſeruauimus, ſi fuerit $u = \text{cof.}\,\omega + \sqrt{-1}\,\text{ſin.}\,\omega$, tum prodire

$$u^n - u^{-n} = 2\sqrt{-1}\,\text{ſin.}\,\omega,$$

 quem

quem ergo valorem nihilo aequalem esse oportet, vnde manente ϱ charactere anguli recti, quia omnium horum angulorum 2ϱ, 4ϱ, 6ϱ, 8ϱ, et in genere $2i\varrho$, sinus euanescunt, idonei anguli pro ω accipiendi hanc progressionem constituent: $\frac{2\varrho}{n}$; $\frac{4\varrho}{n}$; $\frac{6\varrho}{n}$; $\frac{8\varrho}{n}$; etc. quorum quidem numerus est infinitus; verum ob rationes iam ante allegatas hic tantum priores, numero $n-1$, accipi conueniet, ita vt vltimus valorum idoneorum futurus sit $\frac{2(n-1)\varrho}{n}$.

§. 18. Hic quidem videtur, primo loco statui debuisse angulum $\frac{0\varrho}{n}$; verum hinc nasceretur factor $u-2+u^{-1}$, qui inuoluit quadratum, dum est $\frac{1}{u}(u-1)^2$, eiusque radix $u-1$, quae his casibus tantum accipi debet, iam in nostro factore primo $u-u^{-1}$ continetur. Praeterea vero in isto factore primo etiam continetur $u+1$, qui autem in valore nostrum ordinem proxime sequente, qui esset $\frac{2n\varrho}{n}=2\varrho=\omega$, contineretur, siquidem hinc resultaret factor

$$u+2+u^{-1}=\frac{1}{u}(u+1)^2.$$

Caeterum vero patet, si angulos illos pro ω datos vltra terminum praescriptum continuare vellemus, eorum cosinus iam in antecedentibus contineri. Veluti si sumeremus $\omega=\frac{2(n+1)\varrho}{n}$, foret $4\varrho-\omega=\frac{2(n-1)\varrho}{n}$, qui in assignatis valoribus iam continetur.

§. 19. Restituamus nunc loco u valorem assumtum $\cos.\Phi-\sqrt{-1}\sin.\Phi$, ac pro primo factore habebimus $u-u^{-1}=2\sqrt{-1}\sin.\Phi$. Deinde vero pro quolibet factore reliquorum fiet

$$u-2\cos.\omega+u^{-1}=2(\cos.\Phi-\cos.\omega),$$

quamobrem si loco ω successiue valores debitos scribamus,

forma

forma propofita $u^n - u^{-n}$ fequenti producto aequabitur;

$$2^n \sqrt{-1}\, \text{fin.}\, \phi \left(\text{cof.}\, \phi - \text{cof.}\, \tfrac{2\varrho}{n}\right) \left(\text{cof.}\, \phi - \text{cof.}\, \tfrac{4\varrho}{n}\right) \times$$
$$\times \left(\text{cof.}\, \phi - \text{cof.}\, \tfrac{6\varrho}{n}\right) \left(\text{cof.}\, \phi - \text{cof.}\, \tfrac{8\varrho}{n}\right),$$

quorum factorum vltimus erit $\text{cof.}\, \phi - \text{cof.}\, \left(\frac{2(n-1)\varrho}{n}\right)$.

§. 20. Cum igitur fit $\text{fin.}\, n\phi = \frac{1}{2\sqrt{-1}}(u^n - u^{-n})$, hoc valore, quem modo inuenimus, fubftituto, reperiemus:

$$\text{fin.}\, n\phi = 2^{n-1}\, \text{fin.}\, \phi \left(\text{cof.}\, \phi - \text{cof.}\, \tfrac{2\varrho}{n}\right) \left(\text{cof.}\, \phi - \text{cof.}\, \tfrac{4\varrho}{n}\right) \times$$
$$\times \left(\text{cof.}\, \phi - \text{cof.}\, \tfrac{6\varrho}{n}\right) \;.\;.\;.\; \left(\text{cof.}\, \phi - \text{cof.}\, \tfrac{2(n-1)\varrho}{n}\right),$$

vnde fequentes cafus fpeciales deriuaffe iuuabit:

$\text{fin.}\, \phi = \text{fin.}\, \phi.$

$\text{fin.}\, 2\phi = 2\, \text{fin.}\, \phi\, \text{cof.}\, \phi.$

$\text{fin.}\, 3\phi = 4\, \text{fin.}\, \phi (\text{cof.}\, \phi - \text{cof.} \tfrac{2}{3}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{4}{3}\varrho).$

$\text{fin.}\, 4\phi = 8\, \text{fin.}\, \phi (\text{cof.}\, \phi - \text{cof.} \tfrac{2}{4}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{4}{4}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{6}{4}\varrho).$

$$\text{fin.}\, 5\phi = 16\, \text{fin.}\, \phi (\text{cof.}\, \phi - \text{cof.} \tfrac{2}{5}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{4}{5}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{6}{5}\varrho) \times$$
$$\times (\text{cof.}\, \phi - \text{cof.} \tfrac{8}{5}\varrho).$$

$$\text{fin.}\, 6\phi = 32\, \text{fin.}\, \phi (\text{cof.}\, \phi - \text{cof.} \tfrac{2}{6}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{4}{6}\varrho)(\text{cf.}\, \phi - \text{cf.} \tfrac{6}{6}\varrho) \times$$
$$\times (\text{cof.}\, \phi - \text{cof.} \tfrac{8}{6}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{10}{6}\varrho).$$

$$\text{fin.}\, 7\phi = 64\, \text{fin.}\, \phi (\text{cof.}\, \phi - \text{cof.} \tfrac{2}{7}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{4}{7}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{6}{7}\varrho) \times$$
$$\times (\text{cof.}\, \phi - \text{cof.} \tfrac{8}{7}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{10}{7}\varrho) \times$$
$$\times (\text{cof.}\, \phi - \text{cof.} \tfrac{12}{7}\varrho).$$

$$\text{fin.}\, 8\phi = 128\, \text{fin.}\, \phi (\text{cof.}\, \phi - \text{cof.} \tfrac{2}{8}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{4}{8}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{6}{8}\varrho) \times$$
$$\times (\text{cof.}\, \phi - \text{cof.} \tfrac{8}{8}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{10}{8}\varrho) \times$$
$$\times (\text{cof.}\, \phi - \text{cof.} \tfrac{12}{8}\varrho)(\text{cof.}\, \phi - \text{cof.} \tfrac{14}{8}\varrho).$$

etc. etc.

§. 21. Sepofito iam primo factore fin. Φ, in reliquis, quorum numerus eft $n - 1$, idem vfu venit, quod ante, vt fcilicet illi factores ab extremis aequidiftantes in vnum contrahi queant. Cum enim vltimus factor fit cof. $\frac{2(n-1)\varrho}{n}$, ob angulum $\frac{2(n-1)\varrho}{n} = 2\varrho - \frac{2\varrho}{n}$, eius cofinus erit $= -$ cof. $\frac{2\varrho}{n}$, ita vt ifte factor fit cof. $\Phi +$ cof. $\frac{2\varrho}{n}$, qui ergo per primum multiplicatus dat productum cof. $\Phi^2 - ($cof. $\frac{2\varrho}{n})^2$, Simili modo factores fecundus et penultimus contrahentur in hoc productum: cof. $\Phi^2 - ($cof. $\frac{4\varrho}{n})^2$, hocque modo factorum numerus ad dimidium reuocabitur, fiquidem $n - 1$ fuerit numerus par, ideoque n impar; cafibus autem, quibus $n - 1$ eft impar, ideoque n par, infuper accedet factor medius, qui femper eft cof. Φ; vnde hoc productum ita fe habebit:

$$\text{fin. } n\,\Phi = 2^{n-1}\,\text{fin. }\Phi\,[\text{cof. }\Phi^2 - (\text{cof. }\tfrac{2\varrho}{n})^2]\,[\text{cof. }\Phi^2 - (\text{cof. }\tfrac{4\varrho}{n})^2]\times$$
$$\times\,[\text{cof. }\Phi^2 - \text{cof. }\tfrac{6\varrho}{n})^2]\,[\text{cof. }\Phi^2 - \text{cof. }\tfrac{8\varrho}{n})^2] \text{ etc.}$$

§. 22. Supra autem vidimus effe

$$\text{cof. }\Phi^2 - \text{cof. }\psi^2 = \tfrac{1}{2}\,\text{cof. }2\,\Phi - \tfrac{1}{2}\,\text{cof. }2\,\psi,$$

quae formula denuo refoluitur in hos duos factores:

$$\text{fin. }(\Phi + \psi)\,\text{fin. }(\psi - \Phi),$$

ficque hoc modo omnium factorum numerus erit $n - 1$. Hac igitur refolutione adhibita reperietur:

$$\text{fin. } n\,\Phi = 2^{n-1}\,\text{fin. }\Phi\,\text{fin. }(\tfrac{2\varrho}{n} + \Phi)\,\text{fin. }(\tfrac{2\varrho}{n} - \Phi)\,\text{fin. }(\tfrac{4\varrho}{n} + \Phi)\times$$
$$\times\,\text{fin. }(\tfrac{4\varrho}{n} - \Phi)\,\text{fin. }(\tfrac{6\varrho}{n} + \Phi)\,\text{fin. }(\tfrac{6\varrho}{n} - \Phi) \text{ etc.}$$

fi modo obferuetur cafibus, quibus n eft numerus par, adiungendum effe factorem cof. Φ.

§. 23.

§. 23. Quo indoles huius expreſſionis clarius perſpiciatur, ſequentes caſus ſpeciales euoluamus:

$$\sin.\,\Phi = \sin.\,\Phi.$$

$$\sin.\,2\Phi = 2\sin.\,\Phi\cos.\,\Phi.$$

$$\sin.\,3\Phi = 4\sin.\,\Phi\sin.\,(\tfrac{2}{3}\varrho+\Phi)\sin.\,(\tfrac{2}{3}\varrho-\Phi).$$

$$\sin.\,4\Phi = 8\sin.\,\Phi\sin.\,(\tfrac{2}{4}\varrho+\Phi)\sin.\,(\tfrac{2}{4}\varrho-\Phi)\cos.\,\Phi.$$

$$\sin.\,5\Phi = 16\sin.\,\Phi\sin.\,(\tfrac{2}{5}\varrho+\Phi)\sin.\,(\tfrac{2}{5}\varrho-\Phi)\sin.\,(\tfrac{4}{5}\varrho+\Phi)\times$$
$$\times\sin.\,(\tfrac{4}{5}\varrho-\Phi).$$

$$\sin.\,6\Phi = 32\sin.\,\Phi\sin.\,(\tfrac{2}{6}\varrho+\Phi)\sin.\,(\tfrac{2}{6}\varrho-\Phi)\sin.\,(\tfrac{4}{6}\varrho+\Phi)\sin.\,(\tfrac{4}{6}\varrho-\Phi)\times$$
$$\times\cos.\,\Phi.$$

$$\sin.\,7\Phi = 64\sin.\,\Phi\sin.\,(\tfrac{2}{7}\varrho+\Phi)\sin.\,(\tfrac{2}{7}\varrho-\Phi)\sin.\,(\tfrac{4}{7}\varrho+\Phi)\sin.\,(\tfrac{4}{7}\varrho-\Phi)\times$$
$$\times\sin.\,(\tfrac{6}{7}\varrho+\Phi)\sin.\,(\tfrac{6}{7}\varrho-\Phi).$$

$$\sin.\,8\Phi = 128\sin.\,\Phi\sin.\,(\tfrac{2}{8}\varrho+\Phi)\sin.\,(\tfrac{2}{8}\varrho-\Phi)\sin.\,(\tfrac{4}{8}\varrho+\Phi)\sin.\,(\tfrac{4}{8}\varrho-\Phi)\times$$
$$\times\sin.\,(\tfrac{6}{8}\varrho+\Phi)\sin.\,(\tfrac{6}{8}\varrho-\Phi)\cos.\,\Phi.$$

etc. etc.

§. 24. Vt nunc has formulas rurſus generales reddamus, duos caſus conſtitui conuenit, prouti numerus n fuerit vel par vel impar.

I. Sit $n = 2i$, erit

$$\sin.\,2i\Phi = 2^{2i-1}\sin.\,\Phi\cos.\,\Phi\sin.\,(\tfrac{\varrho}{i}+\Phi)\sin.\,(\tfrac{\varrho}{i}-\Phi)\times$$
$$\times\sin.\,(\tfrac{2\varrho}{i}+\Phi)\sin.\,(\tfrac{2\varrho}{i}-\Phi)\sin.\,(\tfrac{3\varrho}{i}+\Phi)\sin.\,(\tfrac{3\varrho}{i}-\Phi\times$$
$$\times\ldots\ \sin.\,(\tfrac{(i-1)\varrho}{i}+\Phi)\sin.\,\tfrac{(i-1)\varrho}{i}-\Phi).$$

II. Sit

II. Sit $n = 2i + 1$, erit

$$\text{fin.}\,(2i+1)\,\phi = 2^{2i}\,\text{fin.}\,\phi\;\text{fin.}\,(\tfrac{2\varrho}{2i+1} + \phi)\;\text{fin.}\,(\tfrac{2\varrho}{2i+1} - \phi)\;\times$$
$$\times\;\text{fin.}\,(\tfrac{4\varrho}{2i+1} + \phi)\;\text{fin.}\,(\tfrac{4\varrho}{2i+1} - \phi)\;\text{fin.}\,(\tfrac{6\varrho}{2i+1} + \phi)\;\times$$
$$\times\;\text{fin.}\,(\tfrac{6\varrho}{2i+1} - \phi)\;.\;.\;\text{fin.}\,(\tfrac{2i\varrho}{2i+1} + \phi)\;\text{fin.}\,(\tfrac{2i\varrho}{2i+1} - \phi).$$

§. 25. Quoniam hic omnes factores praeter cof. ϕ funt finus, eos, fi lubet, in cofinus conuertere poffumus, ope formulae fin. $\psi = \text{cof.}\,(\varrho - \psi)$, ac pro binis cafibus principalibus fequentes expreffiones reperiemus:

Si $n = 2i$, erit

$$\text{fin.}\,2i\,\phi = 2^{2i-1}\,\text{fin.}\,\phi\;\text{cof.}\,\phi\;\text{cof.}\,(\tfrac{\varrho}{i} + \phi)\;\text{cof.}\,(\tfrac{\varrho}{i} - \phi)\;\times$$
$$\times\;\text{cof.}\,(\tfrac{2\varrho}{i} + \phi)\;\text{cof.}\,(\tfrac{2\varrho}{i} - \phi)\;\text{cof.}\,(\tfrac{3\varrho}{i} + \phi)\;\times$$
$$\times\;\text{cof.}\,(\tfrac{3\varrho}{i} - \phi)\;\text{cof.}\,(\tfrac{4\varrho}{i} + \phi)\;\text{cof.}\,(\tfrac{4\varrho}{i} - \phi)\;\times$$
$$\times\;.\;.\;.\;.\;\text{cof.}\,(\tfrac{(i-1)\varrho}{i} + \phi)\;\text{cof.}\,(\tfrac{(i-1)\varrho}{i} - \phi).$$

Si $n = 2i + 1$, erit

$$\text{fin.}\,(2i+1)\,\phi = 2^{2i}\,\text{fin.}\,\phi\;\text{cof.}\,(\tfrac{\varrho}{2i+1} + \phi)\;\text{cof.}\,(\tfrac{\varrho}{2i+1} - \phi)\;\times$$
$$\times\;\text{cof.}\,(\tfrac{3\varrho}{2i+1} + \phi)\;\text{cof.}\,(\tfrac{3\varrho}{2i+1} - \phi)\;\text{cof.}\,(\tfrac{5\varrho}{2i+1} + \phi)\;\times$$
$$\times\;\text{cof.}\,(\tfrac{5\varrho}{2i+1} - \phi)\;\;.\;\;.\;\;.\;\;.\;\;\text{cof.}\,(\tfrac{(2i-1)\varrho}{2i+1} + \phi)\;\times$$
$$\times\;\text{cof.}\,(\tfrac{(2i-1)\varrho}{2i+1} - \phi).$$

§. 26. Cum igitur duplici modo anguli multipli $n\,\phi$ tam cofinum quam finum per factores fecundum finus fiue cofinus procedentes exhibuerimus, hic iftas expreffiones inuentas coniunctim ante oculos exponamus:

I. Si

I. Si fuerit $n = 2\,i$, erit

Pro coſinu:

$$1^\circ.\ \text{coſ.}\,2\,i\,\Phi = 2^{2i-1}\,\text{ſin.}\left(\tfrac{\rho}{2i}+\Phi\right)\text{ſin.}\left(\tfrac{\rho}{2i}-\Phi\right)\text{ſin.}\left(\tfrac{3\rho}{2i}+\Phi\right)\times$$
$$\times\,\text{ſin.}\left(\tfrac{3\rho}{2i}-\Phi\right)\text{ſin.}\left(\tfrac{5\rho}{2i}+\Phi\right)\text{ſin.}\left(\tfrac{5\rho}{2i}-\Phi\right)\times$$
$$\times \ldots\ldots\ \text{ſin.}\left(\tfrac{(2i-1)\rho}{2i}+\Phi\right)\text{ſin.}\left(\tfrac{(2i-1)\rho}{2i}-\Phi\right).$$

$$2^\circ.\ \text{coſ.}\,2\,i\,\Phi = 2^{2i-1}\,\text{coſ.}\left(\tfrac{\rho}{2i}+\Phi\right)\text{coſ.}\left(\tfrac{\rho}{2i}-\Phi\right)\text{coſ.}\left(\tfrac{3\rho}{2i}+\Phi\right)\times$$
$$\times\,\text{coſ.}\left(\tfrac{3\rho}{2i}-\Phi\right)\text{coſ.}\left(\tfrac{5\rho}{2i}+\Phi\right)\text{coſ.}\left(\tfrac{5\rho}{2i}-\Phi\right)\times$$
$$\times \ldots\ldots\ \text{coſ.}\left(\tfrac{(2i-1)\rho}{2i}+\Phi\right)\text{coſ.}\left(\tfrac{(2i-1)\rho}{2i}-\Phi\right).$$

Pro ſinu:

$$1^\circ.\ \text{ſin.}\,2\,i\,\Phi = 2^{2i-1}\,\text{ſin.}\,\Phi\,\text{coſ.}\,\Phi\,\text{ſin.}\left(\tfrac{\rho}{i}+\Phi\right)\text{ſin.}\left(\tfrac{\rho}{i}-\Phi\right)\text{ſin.}\left(\tfrac{2\rho}{i}+\Phi\right)\times$$
$$\times\,\text{ſin.}\left(\tfrac{2\rho}{i}-\Phi\right)\text{ſin.}\left(\tfrac{3\rho}{i}+\Phi\right)\text{ſin.}\left(\tfrac{3\rho}{i}-\Phi\right)\times$$
$$\times \ldots\ldots\ \text{ſin.}\left(\tfrac{(i-1)\rho}{i}+\Phi\right)\text{ſin.}\left(\tfrac{(i-1)\rho}{i}-\Phi\right).$$

$$2^\circ.\ \text{ſin.}\,2\,i\,\Phi = 2^{2i-1}\,\text{ſin.}\,\Phi\,\text{coſ.}\,\Phi\,\text{coſ.}\left(\tfrac{\rho}{i}+\Phi\right)\text{coſ.}\left(\tfrac{\rho}{i}-\Phi\right)\times$$
$$\times\,\text{coſ.}\left(\tfrac{2\rho}{i}+\Phi\right)\text{coſ.}\left(\tfrac{2\rho}{i}-\Phi\right)\text{coſ.}\left(\tfrac{3\rho}{i}+\Phi\right)\text{coſ.}\left(\tfrac{3\rho}{i}-\Phi\right)\times$$
$$\times \ldots\ldots\ \text{coſ.}\left(\tfrac{(i-1)\rho}{i}+\Phi\right)\text{coſ.}\left(\tfrac{(i-1)\rho}{i}-\Phi\right).$$

II. Si fuerit $n = 2\,i + 1$, erit

Pro coſinu:

$$1^\circ.\ \text{coſ.}\,(2i+1)\,\Phi = 2^{2i}\,\text{coſ.}\,\Phi\,\text{ſin.}\left(\tfrac{\rho}{2i+1}+\Phi\right)\text{ſin.}\left(\tfrac{\rho}{2i+1}-\Phi\right)\times$$
$$\times\,\text{ſin.}\left(\tfrac{3\rho}{2i+1}+\Phi\right)\text{ſin.}\left(\tfrac{3\rho}{2i+1}-\Phi\right)\text{ſin.}\left(\tfrac{5\rho}{2i+1}+\Phi\right)\times$$
$$\times\,\text{ſin.}\left(\tfrac{5\rho}{2i+1}-\Phi\right)\ldots\text{ſin.}\left(\tfrac{(2i-1)\rho}{2i+1}+\Phi\right)\text{ſin.}\left(\tfrac{(2i-1)\rho}{2i+1}-\Phi\right).$$

$2^o.$ $\cos.(2i+1)\phi = 2^{2i}\cos.\phi\cos.(\frac{2\varrho}{2i+1}+\phi)\cos.(\frac{2\varrho}{2i+1}-\phi)\times$
$\times\cos.(\frac{4\varrho}{2i+1}+\phi)\cos.(\frac{4\varrho}{2i+1}-\phi)\cos.(\frac{6\varrho}{2i+1}+\phi)\times$
$\times\cos.(\frac{6\varrho}{2i+1}-\phi)\ldots\cos.(\frac{2i\varrho}{2i+1}+\phi)\cos.(\frac{2i\varrho}{2i+1}-)\phi.$

Pro sinu:

$1^o.$ $\sin.(2i+1)\phi = 2^{2i}\sin.\phi\sin.(\frac{2\varrho}{2i+1}+\phi)\sin.(\frac{2\varrho}{2i+1}-\phi)\times$
$\times\sin.(\frac{4\varrho}{2i+1}+\phi)\sin.(\frac{4\varrho}{2i+1}-\phi)\sin.(\frac{6\varrho}{2i+1}+\phi)\times$
$\times\sin.(\frac{6\varrho}{2i+1}-\phi)\ldots\sin.(\frac{2i\varrho}{2i+1}+\phi)\sin.(\frac{2i\varrho}{2i+1}-\phi).$

$2^o.$ $\sin.(2i+1)\phi = 2^{2i}\sin.\phi\cos.(\frac{\varrho}{2i+1}+\phi)\cos.(\frac{\varrho}{2i+1}-\phi)\times$
$\times\cos.(\frac{3\varrho}{2i+1}+\phi)\cos.(\frac{3\varrho}{2i+1}-\phi\cos.(\frac{5\varrho}{2i+1}+\phi)\times$
$\times\cos.(\frac{5\varrho}{2i+1}-\phi)\ldots\cos.(\frac{(2i-1)\varrho}{2i+1}+\phi)\cos.\frac{(2i-1)\varrho}{2i+1}-\phi).$

§. 27. Hinc patet, casu quo $n = 2i$, tam cosinum quam sinum per aequales angulos exprimi, vnde si binae formae pro cosinu $2i\phi$ inuentae per se inuicem diuidantur, nascetur ista aequatio:

$$1 = \mathrm{tang.}(\tfrac{\varrho}{2i}+\phi)\,\mathrm{tang.}(\tfrac{\varrho}{2i}-\phi)\,\mathrm{tang.}(\tfrac{3\varrho}{2i}+\phi)\times$$
$$\times\,\mathrm{tang.}(\tfrac{3\varrho}{2i}-\phi)\,\mathrm{tang.}(\tfrac{5\varrho}{2i}+\phi)\,\mathrm{tang.}(\tfrac{5\varrho}{2i}-\phi)\times$$
$$\times\ldots\ldots\mathrm{tang.}(\tfrac{(2i-1)\varrho}{2i}+\phi)\,\mathrm{tang.}(\tfrac{(2i-1)\varrho}{2i}-\phi).$$

Hinc igitur sequentes casus speciales deducuntur:

Si $i = 1$, erit

$$1 = \mathrm{tang.}(\tfrac{1}{2}\varrho+\phi)\,\mathrm{tang.}(\tfrac{1}{2}\varrho-\phi).$$

Si $i = 2$, erit

$$1 = \mathrm{tang.}(\tfrac{1}{4}\varrho+\phi)\,\mathrm{tang.}(\tfrac{1}{4}\varrho-\phi)\,\mathrm{tang.}(\tfrac{3}{4}\varrho+\phi)\,\mathrm{tang.}(\tfrac{3}{4}\varrho-\phi).$$

Si $i = 3$, erit

1 =

$$1 = \text{tang.}\,(\tfrac{1}{6}\varrho + \Phi)\,\text{tang.}\,(\tfrac{1}{6}\varrho - \Phi)\,\text{tang.}\,(\tfrac{3}{6}\varrho + \Phi)\,\times$$
$$\times\,\text{tang.}\,(\tfrac{3}{6}\varrho - \Phi)\,\text{tang.}\,(\tfrac{5}{6}\varrho + \Phi)\,\text{tang.}\,(\tfrac{5}{6}\varrho - \Phi).$$

etc. etc.

Quae aequalitas quidem ſponte in oculos incurrit, quoniam productum ex primo factore in vltimum eſt $= 1$, ſimilique modo productum ex ſecundo in penultimum $= 1$, et ita porro, ſiquidem bini tales anguli iunctim ſumpti conficiunt angulum rectum ϱ.

§. 28. Diuidamus ſimili modo binos valores pro ſinu $2\,i\,\Phi$ inuentos, priorem per poſteriorem, ac reſultabit iſta aequalitas:

$$1 = \text{tang.}\,(\tfrac{\varrho}{i} + \Phi)\,\text{tang.}\,(\tfrac{\varrho}{i} - \Phi)\,\text{tang.}\,(\tfrac{2\varrho}{i} + \Phi)\,\text{tang.}\,(\tfrac{2\varrho}{i} - \Phi)\,\times$$
$$\times\,\text{tang.}\,(\tfrac{3\varrho}{i} + \Phi)\,\text{tang.}\,(\tfrac{3\varrho}{i} - \Phi)\;.\;.\;.\;.\;.\;.\;\times$$
$$\times\,\text{tang.}\,(\tfrac{(i-1)\varrho}{i} + \Phi)\,\text{tang.}\,(\tfrac{(i-1)\varrho}{i} - \Phi),$$

vnde ſequuntur hi caſus ſpeciales:

Si $i = 1$, erit

$$1 = \text{tang.}\,(\varrho + \Phi)\,\text{tang.}\,(\varrho - \Phi).$$

Si $i = 2$, erit

$$1 = \text{tang.}\,(\tfrac{1}{2}\varrho + \Phi)\,\text{tang.}\,(\tfrac{1}{2}\varrho - \Phi).$$

Si $i = 3$, erit

$$1 = \text{tang.}\,(\tfrac{1}{3}\varrho + \Phi)\,\text{tang.}\,(\tfrac{1}{3}\varrho - \Phi)\,\text{tang.}\,(\tfrac{2}{3}\varrho + \Phi)\,\text{tang.}\,(\tfrac{2}{3}\varrho - \Phi).$$

etc. etc.

Vbi iterum ratio per ſe eſt manifeſta, cum in genere ſit $\text{tang.}\,\psi\;\text{tang.}\,(\varrho - \psi) = 1$, ob $\text{tang.}\,(\varrho - \psi) = \text{cot.}\,\psi$.

III. Resolutio tangentis anguli $n\,\Phi$ in factores.

§. 29. Ista resolutio peculiari Analysi non eget, propterea quod tang. $n\,\Phi = \frac{\sin.\, n\,\Phi}{\cos.\, n\,\Phi}$, quamobrem tantum opus est, vt singuli factores ipsius sin. $n\,\Phi$ diuidantur per singulos factores ipsius cos. $n\,\Phi$. Cum igitur tam pro sinu quam cosinu duplicem dederimus solutionem, ea combinatione vti conueniet, quae simplicissimam solutionem suppeditat. Hic autem iterum duos casus a se inuicem distinxisse iuuabit, prouti numerus n fuerit vel par vel impar.

I. Casus quo $n = 2\,i$.

§. 30. Hic primo casus speciales euoluamus, tribuendo numero n valores 1, 2, 3 etc. eritque

$$\text{tang. } 2\,\Phi = \frac{\text{sin. }\Phi \text{ cos. }\Phi}{\text{cos. }(\tfrac{1}{2}\,\varrho - \Phi)\text{ cos. }(\tfrac{1}{2}\,\varrho + \Phi)},$$

$$\text{tang. } 4\,\Phi = \frac{\text{sin. }\Phi \text{ cos. }\Phi \text{ cos. }(\tfrac{3}{4}\,\varrho + \Phi)\text{ cos. }(\tfrac{3}{4}\,\varrho - \Phi)}{\text{cos.}(\tfrac{1}{4}\varrho + \Phi)\text{ cos.}(\tfrac{1}{4}\varrho - \Phi)\text{ cos.}(\tfrac{3}{4}\varrho + \Phi)\text{ cos.}(\tfrac{3}{4}\varrho - \Phi)}.$$

Cum igitur hic nulla contractio locum inueniat, has expressiones vlterius continuare superfluum foret; at quando n est numerus impar, ob contractionem formulae prodeunt notatu maxime dignae

II. Casus quo $n = 2\,i + 1$.

§. 31. Loco i ergo sumamus successiue numeros 0, 1, 2, 3, etc., ac primo quidem:

Si $i = 0$ et $n = 1$, erit

$$\text{tang. }\Phi = \frac{\sin.\,\Phi}{\cos.\,\Phi}.$$

Si

Si $i = 1$ et $n = 3$, erit

$$\text{tang. } 3\,\Phi = \text{tang. }\Phi \text{ tang. }(\tfrac{2}{3}\varrho + \Phi) \text{ tang. }(\tfrac{2}{3}\varrho - \Phi).$$

Si $i = 2$ et $n = 5$, erit

$$\text{tang. } 5\,\Phi = \text{tang. }\Phi \text{ tang. }(\tfrac{2}{5}\varrho + \Phi) \text{ tang. }(\tfrac{2}{5}\varrho - \Phi) \times$$
$$\times \text{ tang. }(\tfrac{4}{5}\varrho + \Phi) \text{ tang. }(\tfrac{4}{5}\varrho - \Phi).$$

et generaliter, pro quocunque numero impari $2n + 1$, erit

$$\text{tang. }(2i+1)\Phi = \text{tang. }\Phi \text{ tang. }(\tfrac{2\varrho}{2i+1} + \Phi) \text{ tang. }(\tfrac{2\varrho}{2i+1} - \Phi) \times$$
$$\times \text{ tang. }(\tfrac{4\varrho}{2i+1} + \Phi) \text{ tang. }(\tfrac{4\varrho}{2i+1} - \Phi) \times$$
$$\times \text{ tang. }(\tfrac{6\varrho}{2i+1} + \Phi) \text{ tang. }(\tfrac{6\varrho}{2i+1} - \Phi) \times$$
$$\times \;\ldots\; \text{ tang. }(\tfrac{2i\varrho}{2i+1} + \Phi) \text{ tang. }(\tfrac{2i\varrho}{2i+1} - \Phi).$$

§. 32. Hic igitur egregia theoremata Trigonometrica deducimus. Scilicet ex angulo $3\,\Phi$ habemus:

$$\text{tang. } 3\,\Phi = \text{tang. }\Phi \text{ tang. }(60^\circ + \Phi) \text{ tang. }(60^\circ - \Phi),$$

feu quia tang. $60 + \omega = \text{cot. }(30 - \omega)$, erit

$$\text{tang. } 3\,\Phi = \text{tang. }\Phi \text{ cot. }(30^\circ - \Phi) \text{ cot. }(30^\circ + \Phi),$$

vnde colligitur

$$\text{tang. } 3\,\Phi \text{ tang. }(30^\circ - \Phi) \text{ tang. }(30^\circ + \Phi) = \text{tang. }\Phi.$$

Veluti fi fuerit $\Phi = 20^\circ$, habebimus fequentes relationes:

$$\text{tang. } 60^\circ = \text{tang. } 20^\circ \times \text{tang. } 80^\circ \times \text{tang. } 40^\circ, \text{ ideoque}$$
$$\text{tang. } 60^\circ \times \text{tang. } 10^\circ = \text{tang. } 20^\circ \times \text{tang. } 40^\circ,$$

hinc fequentem proportionem deducimus

$$\text{tang. } 10^\circ : \text{tang. } 20^\circ = \text{tang. } 40^\circ : \text{tang. } 60^\circ,$$

ideoque logarithmos fumendo erit

$$l \text{ tang. } 10^\circ + l \text{ tang. } 60^\circ = l \text{ tang. } 20^\circ + l \text{ tang. } 40^\circ.$$

Ex tabulis autem eſt

l tang. $10° = 9{,}2463188$	l tang. $20° = 9{,}5610659$
l tang. $60 = 10{,}2385606$	l tang. $40 = 9{,}9238135$
l tg. $10° + l$ tg. $60° = 9{,}4848794$	l tg. $20° + l$ tg. $40° = 9{,}4848794$.

§. 33. Simili modo formula pro tang. $5\,\Phi$ inuenta praebet

$$\text{tang.}\,5\,\Phi = \text{tang.}\,\Phi\ \text{tang.}\,(36° + \Phi)\ \text{tang.}\,(36° - \Phi) \times$$
$$\times\ \text{tang.}\,(72° + \Phi)\ \text{tang.}\,(72° - \Phi),\ \text{ſiue}$$

$$\text{tang.}\,5\,\Phi\ \text{tang.}\,(18° + \Phi)\ \text{tang.}\,(18° - \Phi) = \text{tang.}\,\Phi \times$$
$$\times\ \text{tang.}\,(36° + \Phi)\ \text{tang.}\,(36° - \Phi),$$

hincque in logarithmis:

$$l\,\text{tang.}\,5\,\Phi + l\,\text{tang.}\,(18° + \Phi) + l\,\text{tang.}\,(18° - \Phi)$$
$$= l\,\text{tang.}\,\Phi + l\,\text{tang.}\,(36° + \Phi) + l\,\text{tang.}\,(36° - \Phi).$$

Sit exempli gratia $\Phi = 10°$, eritque

$$l\,\text{tang.}\,50 + l\,\text{tang.}\,28° + l\,\text{tang.}\,8°$$
$$= l\,\text{tang.}\,10° + l\,\text{tang.}\,46° + l\,\text{tang.}\,26°,$$

veluti in ſequente ſchematismo videre licet:

l tang. $50° = 10{,}0761865$	l tang. $10° = 9{,}2463188$
l tang. $28 = 9{,}7256744$	l tang. $46 = 10{,}0151628$
l tang. $8 = 9{,}1478025$	l tang. $26 = 9{,}6881818$
l tg. $50° + l$ tg. $28° + l$ tg. $8° = 8{,}9496634$	Summa $= 8{,}9496634$.

§. 34. Quoniam igitur hactenus tam ſinus et coſinus quam tangentes angulorum multiplorum per producta expresſimus, iſtas formulas commode per logarithmos euoluere licebit, qui deinceps per differentiationem tolli poterunt, dum ſcilicet angulus Φ tanquam variabilis ſpectatur.

Euolu-

Euolutio

formularum pro cof. $n\,\Phi$ inuentarum, per logarithmos et differentiationem.

§. 35. Cum fupra inuenerimus

$$\operatorname{cof.} 2\,\Phi = 2 \operatorname{cof.} (\tfrac{1}{2}\varrho + \Phi) \operatorname{cof.} (\tfrac{1}{2}\varrho - \Phi),$$

erit fumtis logarithmis

$$l \operatorname{cof.} 2\,\Phi = l\,2 + l \operatorname{cof.} (\tfrac{1}{2}\varrho + \Phi) + l \operatorname{cof.} (\tfrac{1}{2}\varrho - \Phi),$$

quae aequatio quia vera eft pro angulo quocunque Φ, fpectetur Φ vt quantitas variabilis, ac differentiatio nobis praebebit hanc aequationem:

$$-\frac{2\,\partial\Phi \operatorname{fin.} 2\,\Phi}{\operatorname{cof.} 2\,\Phi} = -\frac{\partial\Phi \operatorname{fin.} (\tfrac{1}{2}\varrho + \Phi)}{\operatorname{cof.} (\tfrac{1}{2}\varrho + \Phi)} + \frac{\partial\Phi \operatorname{fin.} (\tfrac{1}{2}\varrho - \Phi)}{\operatorname{cof.} (\tfrac{1}{2}\varrho - \Phi)},$$

quae per $-\partial\Phi$ diuifa per tangentes dabit

$$2 \operatorname{tang.} 2\,\Phi = \operatorname{tang.} (45^\circ + \Phi) - \operatorname{tang.} (45^\circ - \Phi),$$

id quod exemplo illuftraffe iuuabit. Sit igitur $\Phi = 17^\circ, 30'$, eritque

$$2 \operatorname{tang.} 35^\circ = \operatorname{tang.} (62^\circ, 30') - \operatorname{tang.} (27^\circ, 30').$$

Eft vero ex tabulis

	tang. $62^\circ, 30'$	$= 1,9209821$
	tang. $27, 30$	$= 0,5205671$
$2 \operatorname{tang.} 35^\circ = 1,4004150$	Differentia	$= 1,4004150$

§. 36. Deinde quia fupra inuenimus

$$\operatorname{cof.} 3\,\Phi = 4 \operatorname{cof.} (60^\circ + \Phi) \operatorname{cof.} (60^\circ - \Phi) \operatorname{cof.} \Phi,$$

erit per logarithmos

$$l \operatorname{cof.} 3\,\Phi = l\,4 + l \operatorname{cof.} \Phi + l \operatorname{cof.} (60^\circ + \Phi)\; l \operatorname{cof.} (60^\circ - \Phi).$$

Hinc

Hinc differentiando erit

$$3 \text{ tang. } 3\phi = \text{tang. } \phi + \text{tang. } (60^\circ + \phi) - \text{tang. } (60 - \phi).$$

Exemplum. Sumatur $\phi = 25^\circ$, eritque

$$3 \text{ tang. } 75^\circ = \text{tang. } 25^\circ + \text{tang. } 85^\circ - \text{tang. } 35^\circ,$$

cuius veritas ex ſubiecto calculo videre licet:

$$\begin{array}{r|l}
 & \text{tang. } 25^\circ = 0,4663077 \\
 & \text{tang. } 85. = 11,4300520 \\
\hline
 & \text{Summa} = 11,8963597 \\
 & \text{tang. } 35^\circ = 0,7002075 \\
\hline
3 \text{ tang. } 75^\circ = 11,1961524 & \phantom{\text{tang. } 35^\circ =}\ 11,1961522.
\end{array}$$

Aliud exemplum. Sumamus $\phi = 29^\circ$, eritque

$$3 \text{ tang. } 87^\circ = \text{tang. } 29^\circ + \text{tang. } 89^\circ - \text{tang. } 31^\circ,$$

veluti ex ſubiecto calculo videre licet

$$\begin{array}{r|l}
\text{tang. } 87^\circ = 19,081137 & \text{tang. } 29^\circ = 0,554309 \\
\hline
3 \text{ tang. } 87^\circ = 57,243411 & \text{tang. } 89 = 57,289962 \\
\hline
 & \text{Summa} = 57,844271 \\
 & \text{tang. } 31^\circ = 0,600860 \\
\hline
 & \text{different.} = 57,243411.
\end{array}$$

§. 37. Sumamus etiam exemplum quo anguli occurrunt maiores recto, ſitque $\phi = 58^\circ$, eſſeque oportet:

$$3 \text{ tang. } 174^\circ = \text{tang. } 58^\circ + \text{tang. } 118^\circ - \text{tang. } 2^\circ,$$

quae aequatio reducitur ad ſequentem:

$$3 \text{ tang. } 6^\circ = \text{tang. } 62^\circ + \text{tang. } 2^\circ - \text{tang. } 58^\circ,$$

et calculus ita ſe habebit:

tang.

tang. $6^\circ = 0,1051042$	tang. $62^\circ = 1,8807265$
3 tang. $6^\circ = 0,3153126$	tang. $2^\circ = 0,0349208$
	Summa $= 1,9156473$
	tang. $38^\circ = 1,6003345$
	different. $= 0,3153128.$

§. 38. Sit nunc in genere $n = 2i$, et quia inuenimus

$$\operatorname{cof.} 2i\Phi = 2^{2i-1} \operatorname{cof.} (\tfrac{\rho}{2i} + \Phi) \operatorname{cof.} (\tfrac{\rho}{2i} - \Phi) \operatorname{cof.} (\tfrac{3\rho}{2i} + \Phi) \times$$
$$\times \operatorname{cof.} (\tfrac{3\rho}{2i} - \Phi) \operatorname{cof.} (\tfrac{5\rho}{2i} + \Phi) \ldots \operatorname{cof.} (\tfrac{(2i-1)\rho}{2i} + \Phi) \times$$
$$\times \operatorname{cof.} (\tfrac{(2i-1)\rho}{2i} - \Phi),$$

hinc per logarithmos erit

$$l \operatorname{cof.} 2i\Phi = l\, 2^{2i-1} + l \operatorname{cof.} (\tfrac{\rho}{2i} + \Phi) + l \operatorname{cof.} (\tfrac{\rho}{2i} - \Phi)$$
$$+ l \operatorname{cof.} (\tfrac{3\rho}{2i} + \Phi) + \operatorname{cof.} l (\tfrac{3\rho}{2i} - \Phi) \ldots$$
$$+ l \operatorname{cof.} (\tfrac{(2i-1)\rho}{2i} + \Phi) + l \operatorname{cof.} (\tfrac{(2i-1)\rho}{2i} - \Phi),$$

vnde ergo per differentiationem nancifcimur

$$2i \operatorname{tang.} 2i\Phi = \operatorname{tang.} (\tfrac{\rho}{2i} + \Phi) - \operatorname{tang.} (\tfrac{\rho}{2i} - \Phi)$$
$$+ \operatorname{tang.} (\tfrac{3\rho}{2i} + \Phi) - \operatorname{tang.} (\tfrac{3\rho}{2i} - \Phi) \text{ etc.} \ldots$$
$$+ \operatorname{tang.} (\tfrac{(2i-1)\rho}{2i} + \Phi) - \operatorname{tang.} (\tfrac{(2i-1)\rho}{2i} - \Phi).$$

§. 39. Hinc ergo pro i fumendis numeris minoribus habebimus:

$$4 \operatorname{tang.} 4\Phi = \operatorname{tang.} (22\tfrac{1}{2}^\circ + \Phi) \operatorname{tang.} (22\tfrac{1}{2}^\circ - \Phi)$$
$$+ \operatorname{tang.} (67\tfrac{1}{2}^\circ + \Phi) - \operatorname{tang.} (67\tfrac{1}{2}^\circ - \Phi);$$
$$6 \operatorname{tang.} 6\Phi = \operatorname{tang.} (15^\circ + \Phi) - \operatorname{tang.} (15^\circ - \Phi)$$
$$+ \operatorname{tang.} (45^\circ + \Phi) - \operatorname{tang.} (45^\circ - \Phi)$$
$$+ \operatorname{tang.} (75^\circ + \Phi) - \operatorname{tang.} (75^\circ - \Phi);$$

$$8 \text{ tang. } 8\Phi = \text{tang. } (11\tfrac{1}{4}^\circ + \Phi) - \text{tang. } (11\tfrac{1}{4}^\circ - \Phi)$$
$$+ \text{tang. } (33\tfrac{3}{4}^\circ + \Phi) - \text{tang. } (33\tfrac{3}{4}^\circ - \Phi)$$
$$+ \text{tang. } (56\tfrac{1}{4}^\circ + \Phi) - \text{tang. } (56\tfrac{1}{4}^\circ - \Phi)$$
$$+ \text{tang. } (78\tfrac{3}{4}^\circ + \Phi) - \text{tang. } (78\tfrac{3}{4}^\circ - \Phi).$$

§. 40. Sumamus nunc etiam $n = 2i + 1$, quo casu vidimus esse

$$\text{cos.} (2i+1)\Phi = 2^{2i} \text{cos.} \Phi \text{ cos.} (\tfrac{2\varrho}{2i+1} + \Phi) \text{ cos.} (\tfrac{2\varrho}{2i+1} - \Phi) \times$$
$$\times \text{cos.} (\tfrac{4\varrho}{2i+1} + \Phi) \text{ cos.} (\tfrac{4\varrho}{2i+1} - \Phi) \text{ cos.} (\tfrac{6\varrho}{2i+1} + \Phi) \times$$
$$\times \quad . \quad . \quad . \quad . \quad \text{cos.} (\tfrac{2i\varrho}{2i+1} + \Phi) \text{ cos.} (\tfrac{2i\varrho}{2i+1} - \Phi),$$

hinc sumtis logarithmis erit

$$l \text{ cos.} (2i+1)\Phi = l\, 2^{2i} + l \text{ cos.} \Phi + l \text{ cos.} (\tfrac{2\varrho}{2i+1} + \Phi)$$
$$+ l \text{ cos.} (\tfrac{2\varrho}{2i+1} - \Phi) \; . \; . \; . \; . + l \text{ cos.} (\tfrac{2i\varrho}{2i+1} + \Phi)$$
$$+ l \text{ cos.} (\tfrac{2i\varrho}{2i+1} - \Phi),$$

vnde differentiando nanciscimur

$$(2i+1) \text{ tang.} (2i+1)\Phi = \text{tang.} \Phi + \text{tang.} (\tfrac{2\varrho}{2i+1} + \Phi)$$
$$- \text{tang.} (\tfrac{2\varrho}{2i+1} - \Phi) + \text{tang.} (\tfrac{4\varrho}{2i+1} + \Phi)$$
$$- \text{tang.} (\tfrac{4\varrho}{2i+1} - \Phi) \quad . \quad . \quad . \quad . \quad .$$
$$+ \text{tang.} (\tfrac{2i\varrho}{2i+1} + \Phi) - \text{tang.} (\tfrac{2i\varrho}{2i+1} - \Phi).$$

§. 41. Casum quo $2i + 1 = 3$ iam euoluimus, sit igitur $2i + 1 = 5$, eritque

$$5 \text{ tang. } 5\Phi = \text{tang. } \Phi + \text{tang. } (36^\circ + \Phi) - \text{tang. } (36^\circ - \Phi)$$
$$+ \text{tang. } (72^\circ + \Phi) - \text{tang. } (72^\circ - \Phi).$$

Sit

Sit nunc $2i + 1 = 7$, eritque

$$7 \text{ tang. } 7\phi = \text{tang. } \phi + \text{tang. } (25\tfrac{4}{7}^\circ + \phi)$$
$$- \text{tang. } (25\tfrac{4}{7}^\circ - \phi) + \text{tang. } (51\tfrac{1}{7}^\circ + \phi)$$
$$- \text{tang. } (51\tfrac{1}{7}^\circ - \phi) + \text{tang. } (76\tfrac{5}{7}^\circ + \phi)$$
$$- \text{tang. } (76\tfrac{5}{7}^\circ - \phi).$$

Posito $2i + 1 = 9$ habebimus

$$9 \text{ tang. } 9\phi = \text{tang. } \phi + \text{tang. } (20^\circ + \phi) - \text{tang. } (20^\circ - \phi)$$
$$+ \text{tang. } (40^\circ + \phi) - \text{tang. } (40^\circ - \phi)$$
$$+ \text{tang. } (60^\circ + \phi) - \text{tang. } (60^\circ - \phi)$$
$$+ \text{tang. } (80^\circ + \phi) - \text{tang. } (80^\circ - \phi).$$